

Lepton Family Symmetries for Neutrino Masses and Mixing

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Introduction

$$\mathcal{M}_l = U_L^l \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} (U_R^l)^\dagger,$$
$$\mathcal{M}_\nu^D = U_L^\nu \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} (U_R^\nu)^\dagger,$$
$$\mathcal{M}_\nu^M = U_L^\nu \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} (U_L^\nu)^T,$$

$$\begin{aligned}
U_{l\nu} &= (U_L^l)^\dagger U_L^\nu = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{pmatrix} \\
&\simeq \begin{pmatrix} 0.83 & 0.56 & < 0.2 \\ -0.39 & 0.59 & -0.71 \\ -0.39 & 0.59 & 0.71 \end{pmatrix} \\
&\simeq \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \sim (\eta_8, \eta_1, \pi^0)
\end{aligned}$$

[Harrison/ Perkins/Scott (2002)]

Historically, once the third lepton τ was established, it was speculated by [Cabibbo \[1978\]](#) and [Wolfenstein \[1978\]](#) that

$$U_{l\nu}^{CW} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},$$

where $\omega = \exp(2\pi i/3) = -1/2 + i\sqrt{3}/2$, and

$$U_{l\nu}^{HPS} = (U_{l\nu}^{CW})^\dagger \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}.$$

How can $U_{l\nu}^{HPS}$, i.e. **tribimaximal** mixing, be derived from a symmetry? The difficulty comes from the fact that any symmetry defined in the basis $(\nu_e, \nu_\mu, \nu_\tau)$ is automatically applicable to (e, μ, τ) in the complete Lagrangian. Usually one assumes the canonical seesaw mechanism and studies $\mathcal{M}_\nu = -\mathcal{M}_\nu^D \mathcal{M}_N^{-1} (\mathcal{M}_\nu^D)^T$ in the basis where \mathcal{M}_l is diagonal. But the symmetry apparent in \mathcal{M}_ν is often incompatible with a diagonal \mathcal{M}_l with eigenvalues m_e, m_μ, m_τ . To obtain $U_{l\nu}^{HPS}$, the discrete symmetry A_4 turns out to be useful. In this talk, I will focus mainly on this approach, but will also mention S_4 and $\Delta(27)$.

Some Discrete Symmetries

solid	faces	vertices	Plato	group
tetrahedron	4	4	fire	A_4
octahedron	8	6	air	S_4
cube	6	8	earth	S_4
icosahedron	20	12	water	A_5
dodecahedron	12	20	quintessence	A_5

Compare this to today's **TOE**, i.e. **string theory**. There are 5 consistent theories in 10 dimensions: Type I is dual to Heterotic $SO(32)$, Type IIA is dual to Heterotic $E_8 \times E_8$, and Type IIB is self-dual.

Tetrahedral Symmetry A_4

For 3 families, we should look for a group with a 3 representation, the simplest of which is A_4 , the group of the **even** permutation of 4 objects.

class	n	h	χ_1	$\chi_{1'}$	$\chi_{1''}$	χ_3
C_1	1	1	1	1	1	3
C_2	4	3	1	ω	ω^2	0
C_3	4	3	1	ω^2	ω	0
C_4	3	2	1	1	1	-1

$$\omega = \exp(2\pi i/3) = -1/2 + i\sqrt{3}/2$$

Multiplication rule:

$$\begin{aligned} \underline{3} \times \underline{3} = & \underline{1}(\underline{11} + \underline{22} + \underline{33}) + \underline{1}'(\underline{11} + \omega^2\underline{22} + \omega\underline{33}) \\ & + \underline{1}''(\underline{11} + \omega\underline{22} + \omega^2\underline{33}) + \underline{3}(\underline{23}, \underline{31}, \underline{12}) + \underline{3}(\underline{32}, \underline{13}, \underline{21}). \end{aligned}$$

Note that $\underline{3} \times \underline{3} \times \underline{3} = \underline{1}$ is possible in A_4 ,

i.e. $a_1 b_2 c_3 + \text{permutations}$,

and $\underline{2} \times \underline{2} \times \underline{2} = \underline{1}$ is possible in S_3 ,

i.e. $a_1 b_1 c_1 + a_2 b_2 c_2$.

(I) Ma/Rajasekaran (2001):

$(\nu_i, l_i) \sim \underline{\mathbf{3}}, l_i^c \sim \underline{\mathbf{1}}, \underline{\mathbf{1}'}, \underline{\mathbf{1}''}$, then with $(\phi_i^0, \phi_i^-) \sim \underline{\mathbf{3}}$,

$$\mathcal{M}_l = \begin{pmatrix} h_1 v_1 & h_2 v_1 & h_3 v_1 \\ h_1 v_2 & h_2 \omega v_2 & h_3 \omega^2 v_2 \\ h_1 v_3 & h_2 \omega^2 v_3 & h_3 \omega v_3 \end{pmatrix}$$

$$= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \sqrt{3} v$$

for $v_1 = v_2 = v_3 = v$.

(II) **Ma (2006)**:

$(\nu_i, l_i) \sim \underline{\mathbf{3}}, l_i^c \sim \underline{\mathbf{3}}$, then with $(\phi_i^0, \phi_i^-) \sim \underline{\mathbf{1}}, \underline{\mathbf{3}}$,

$$\mathcal{M}_l = \begin{pmatrix} h_0 v_0 & h_1 v_3 & h_2 v_2 \\ h_2 v_3 & h_0 v_0 & h_1 v_1 \\ h_1 v_2 & h_2 v_1 & h_0 v_0 \end{pmatrix} = [\text{for } v_{1,2,3} = v]$$

$$\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}$$

$m_e = h_0 v_0 + (h_1 + h_2)v$, $m_\mu = h_0 v_0 + (h_1 \omega + h_2 \omega^2)v$,
and $m_\tau = h_0 v_0 + (h_1 \omega^2 + h_2 \omega)v$.

In either case, $U_{l\nu}^{CW}$ has been derived, if mixing occurs only in \mathcal{M}_l . To obtain $U_{l\nu}^{HPS}$ instead, \mathcal{M}_ν must be considered. Let \mathcal{M}_ν be Majorana and come from Higgs triplets, then

$$\mathcal{M}_\nu = \begin{pmatrix} a + b + c & f & e \\ f & a + \omega b + \omega^2 c & d \\ e & d & a + \omega^2 b + \omega c \end{pmatrix}$$

where a comes from $\underline{1}$, b from $\underline{1}'$, c from $\underline{1}''$, and (d, e, f) from $\underline{3}$. To proceed further, the 6 parameters of \mathcal{M}_ν must be restricted.

Selected A_4 Models

The first two proposed A_4 models start with only $a \neq 0$, yielding thus 3 degenerate neutrino masses. In [Ma/Rajasekaran\(2001\)](#), the degeneracy is broken softly by $N_i N_j$ terms, allowing b, c, d, e, f to be nonzero. In [Babu/Ma/Valle\(2003\)](#), the degeneracy is broken radiatively through flavor-changing supersymmetric scalar lepton mass terms. In both cases, $\theta_{23} \simeq \pi/4$ is predicted. In [BMV03](#), maximal CP violation in $U_{l\nu}$ is also predicted.

Ma (2004):

Let $b = c$ and $e = f = 0$, then

$$\mathcal{M}_\nu = \begin{pmatrix} a + 2b & 0 & 0 \\ 0 & a - b & d \\ 0 & d & a - b \end{pmatrix},$$

which is diagonalized by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & i \end{pmatrix},$$

with eigenvalues $a - b + d$, $a + 2b$, $-a + b + d$.

Thus **tribimaximal** mixing is achieved. However, since $\underline{1}'$ and $\underline{1}''$ are unrelated in A_4 , $b = c$ is rather *ad hoc*.

Altarelli/Feruglio (2005):

Eliminate both $\underline{1}'$ and $\underline{1}''$, then $b = c = 0$ implies $m_1 = a + d$, $m_2 = a$, $m_3 = -a + d$.

Ma (2005):

$\Delta m_{sol}^2 \ll \Delta m_{atm}^2$ implies $|d| \simeq -2|a| \cos \phi$,
 $|m_{1,2}|^2 \simeq |a|^2$, $|m_3|^2 \simeq |a|^2(1 + 8 \cos^2 \phi)$,
i.e. normal ordering with

$$|m_{\nu_e}|^2 \simeq \frac{\Delta m_{atm}^2}{8 \cos^2 \phi} \simeq |m_{ee}|^2 + \frac{\Delta m_{atm}^2}{9}.$$

Babu/He (2005):

Let $\mathcal{M}_\nu^D \sim 1$ and

$$\mathcal{M}_N = \begin{pmatrix} A & 0 & 0 \\ 0 & A & D \\ 0 & D & A \end{pmatrix},$$

then \mathcal{M}_ν again has two parameters, with $e = f = 0$,
 $b = c$, $d^2 = 3b(b - a)$.

This scheme is essentially the *inverse* of **AF05** and predicts an inverted ordering of neutrino masses.

Technical challenge is to break A_4 spontaneously along two incompatible directions: $(1, 1, 1)$ and $(0, 0, 1)$. Complicated auxiliary fields and symmetries appear to be necessary for this to happen.

Further, although $b \neq c$ would allow $U_{e3} \neq 0$, the assumption $e = f = 0$ and the bound $|U_{e3}| < 0.16$ imply $0.5 < \tan^2 \theta_{12} < 0.52$.

Experimentally, $\tan^2 \theta_{12} = 0.45 \pm 0.05$.

Approximate Tribimaximal Mixing

(III) Hirsch/Ma/Valle/Villanova (2005):

Let $(\nu_i, l_i), l_i^c \sim \underline{3}$ and $(\phi_i^0, \phi_i^-) \sim \underline{1}, \underline{1}', \underline{1}''$,

then \mathcal{M}_l is diagonal with

$$\begin{pmatrix} m_e \\ m_\mu \\ m_\tau \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} h_1 v_1 \\ h_2 v_2 \\ h_3 v_3 \end{pmatrix}.$$

Here $\mathcal{M}_\nu^{(e,\mu,\tau)} = \mathcal{M}_\nu$ already.

Let $d = e = f$, then

$$\mathcal{M}_\nu = \begin{pmatrix} a + b + c & d & d \\ d & a + \omega b + \omega^2 c & d \\ d & d & a + \omega^2 b + \omega c \end{pmatrix}$$

Assume $b = c$ and rotate to the basis

$[\nu_e, (\nu_\mu + \nu_\tau)/\sqrt{2}, (-\nu_\mu + \nu_\tau)/\sqrt{2}]$, then

$$\mathcal{M}_\nu = \begin{pmatrix} a + 2b & \sqrt{2}d & 0 \\ \sqrt{2}d & a - b + d & 0 \\ 0 & 0 & a - b - d \end{pmatrix},$$

i.e. **maximal $\nu_\mu - \nu_\tau$ mixing** and $U_{e3} = 0$.

We now have $\tan 2\theta_{12} = 2\sqrt{2}d/(d - 3b)$.

To obtain $\tan^2 \theta_{12} = 1/2$, we need $b = 0$ or $b = 2d/3$.

However, $\Delta m_{sol}^2 \ll \Delta m_{atm}^2$ implies $2a + b + d \rightarrow 0$, so that $\Delta m_{atm}^2 \rightarrow 6bd$. Therefore, $b \neq 0$ is required, but should be close to it, because $b = 0$ enhances the symmetry of \mathcal{M}_ν from Z_2 to S_3 .

Here $\tan^2 \theta_{12} < 1/2$ implies inverted ordering, and $\tan^2 \theta_{12} > 1/2$ implies normal ordering.

Of course, we can choose $b = 2d/3$, but it would not be a prediction.

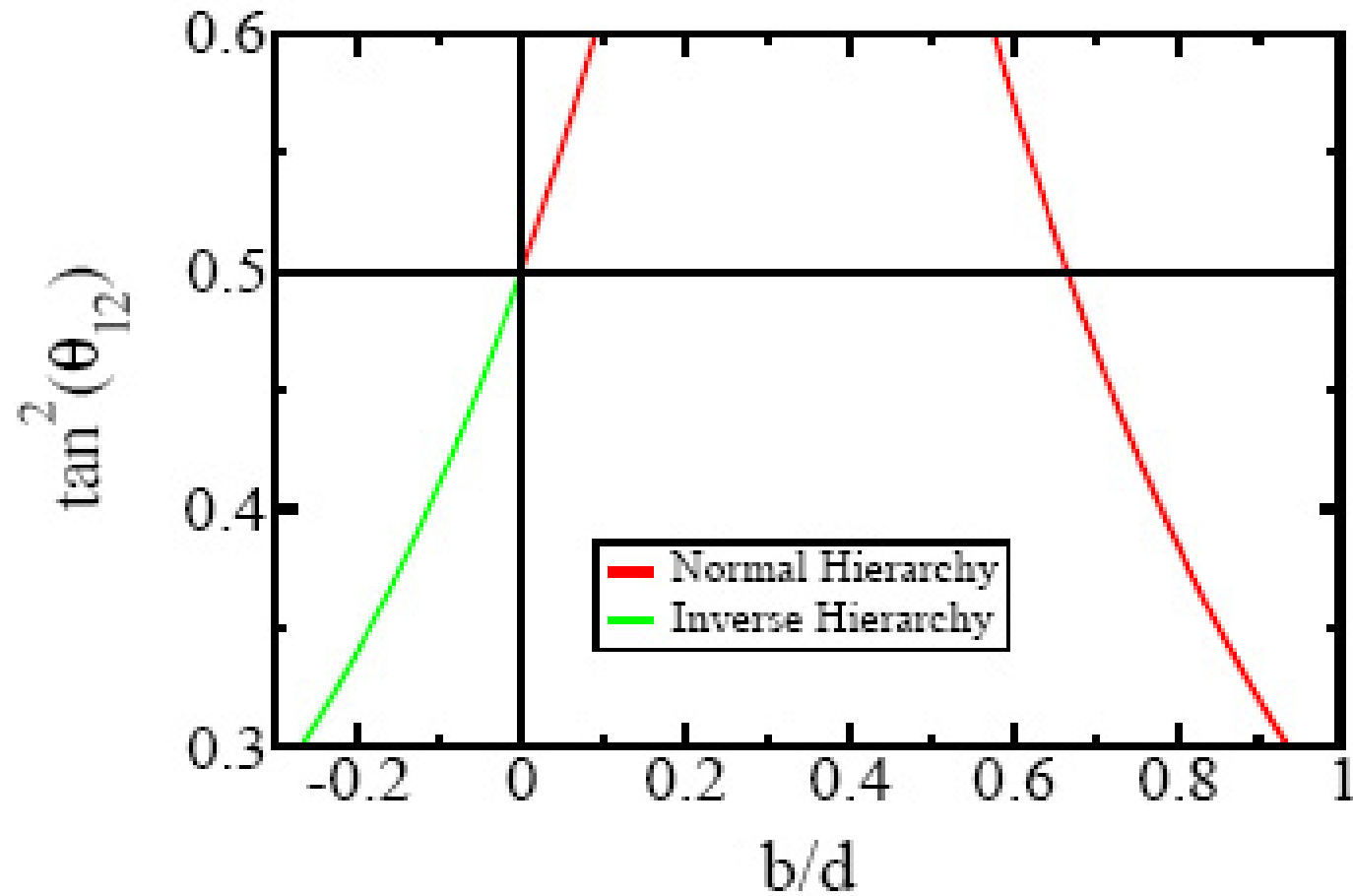


FIG. 1: Solar mixing angle, $\tan^2 \theta_{12}$, vs. b/d .

S_4 and $\Delta(27)$

Some more recent applications are based on S_4 [Ma(2006)] and $\Delta(27)$ [Ma(2006)], resulting in:

$$\mathcal{M}_\nu(S_4) = \begin{pmatrix} a + 2b & e & e \\ e & a - b & d \\ e & d & a - b \end{pmatrix},$$

$$\mathcal{M}_\nu(\Delta(27)) = \begin{pmatrix} fa & c & b \\ c & fb & a \\ b & a & fc \end{pmatrix}.$$

Permutation Symmetry S_4

The group of permutation of 4 objects is S_4 . It contains both S_3 and A_4 . It is also the symmetry group of the hexahedron (cube) and the octahedron.

class	n	h	χ_1	$\chi_{1'}$	χ_2	χ_3	$\chi_{3'}$
C_1	1	1	1	1	2	3	3
C_2	3	2	1	1	2	-1	-1
C_3	8	3	1	1	-1	0	0
C_4	6	4	1	-1	0	-1	1
C_5	6	2	1	-1	0	1	-1

Multiplication rule:

$$\underline{3} \times \underline{3} = \underline{1}(\underline{11} + \underline{22} + \underline{33}) + \underline{2}(\underline{11} + \omega^2 \underline{22} + \omega \underline{33}, \underline{11} + \omega \underline{22} + \omega^2 \underline{33}) \\ + \underline{3}(\underline{23} + \underline{32}, \underline{31} + \underline{13}, \underline{12} + \underline{21}) + \underline{3}'(\underline{23} - \underline{32}, \underline{31} - \underline{13}, \underline{12} - \underline{21})$$

$$\underline{3}' \times \underline{3}' = \underline{1} + \underline{2} + \underline{3}_S + \underline{3}'_A$$

$$\underline{3} \times \underline{3}' = \underline{1}' + \underline{2} + \underline{3}'_S + \underline{3}_A$$

Note that both $\underline{3} \times \underline{3} \times \underline{3} = \underline{1}$ and $\underline{2} \times \underline{2} \times \underline{2} = \underline{1}$ are possible in S_4 .

Let $(\nu_i, l_i), l_i^c, N_i \sim \underline{\mathbf{3}}$.

Assume singlet Higgs superfields $\sigma_{1,2,3} \sim \underline{\mathbf{3}}$ and $\zeta_{1,2} \sim \underline{\mathbf{2}}$, then

$$\mathcal{M}_N = \begin{pmatrix} M_1 & h\langle\sigma_3\rangle & h\langle\sigma_2\rangle \\ h\langle\sigma_3\rangle & M_2 & h\langle\sigma_1\rangle \\ h\langle\sigma_2\rangle & h\langle\sigma_1\rangle & M_3 \end{pmatrix}$$

where

$$M_1 = A + f(\langle\zeta_2\rangle + \langle\zeta_1\rangle)$$

$$M_2 = A + f(\omega\langle\zeta_2\rangle + \omega^2\langle\zeta_1\rangle)$$

$$M_3 = A + f(\omega^2\langle\zeta_2\rangle + \omega\langle\zeta_1\rangle)$$

The most general S_4 -invariant superpotential of σ and ζ

is given by

$$W = M(\sigma_1\sigma_1 + \sigma_2\sigma_2 + \sigma_3\sigma_3) + \lambda\sigma_1\sigma_2\sigma_3 + m\zeta_1\zeta_2 + \rho(\zeta_1\zeta_1\zeta_1 + \zeta_2\zeta_2\zeta_2) + \kappa(\sigma_1\sigma_1 + \omega\sigma_2\sigma_2 + \omega^2\sigma_3\sigma_3)\zeta_2 + \kappa(\sigma_1\sigma_1 + \omega^2\sigma_2\sigma_2 + \omega\sigma_3\sigma_3)\zeta_1$$

The resulting scalar potential has a minimum at $V = 0$ (thus preserving supersymmetry) only if $\langle \zeta_1 \rangle = \langle \zeta_2 \rangle$ and $\langle \sigma_2 \rangle = \langle \sigma_3 \rangle$, so that

$$\mathcal{M}_N = \begin{pmatrix} A + 2B & E & E \\ E & A - B & D \\ E & D & A - B \end{pmatrix}$$

Choose $\mathcal{M}_{\nu N} \propto$ the identity with $(\phi^0, \phi^-) \sim \underline{1}$, then if \mathcal{M}_l is also diagonal (III) with $(\phi^0, \phi^-) \sim \underline{1}, \underline{2}$,

$$\mathcal{M}_\nu = \begin{pmatrix} a + 2b & e & e \\ e & a - b & d \\ e & d & a - b \end{pmatrix}.$$

This avoids the *ad hoc* assumption of equating the contributions of the $\underline{1}'$ and $\underline{1}''$ contributions in A_4 .

Thus $\theta_{13} = 0$ is a *bona fide* prediction.

Now $\tan 2\theta_{12} = 2\sqrt{2}e/(d - 3b)$, and $\Delta m_{sol}^2 \ll \Delta m_{atm}^2$ implies $\Delta m_{atm}^2 \rightarrow 6bd + d^2 - e^2$.

If \mathcal{M}_l is of the form (II) with $(\phi^0, \phi^-) \sim \underline{1}, \underline{3}, \underline{3}'$, then
 $\mathcal{M}_\nu =$

$$\begin{pmatrix} a + (2d + 4e)/3 & b - (d - e)/3 & b - (d - e)/3 \\ b - (d - e)/3 & b + 2(d - e)/3 & a - (d + 2e)/3 \\ b - (d - e)/3 & a - (d + 2e)/3 & b + 2(d - e)/3 \end{pmatrix}.$$

Here $\tan 2\theta_{12} = 2\sqrt{2}(d - 3b - e)/(d - 3b + 8e)$.

Therefore, $\tan^2 \theta_{12} \rightarrow 1/2$ for $e \rightarrow 0$,

with $\tan^2 \theta_{12} < 1/2$ for $e/(d - 3b) > 0$.

$\Delta(3n^2)$ are discrete subgroups of $SU(3)$:

$$\Delta(3) \sim Z_3, \quad \Delta(12) \sim A_4.$$

$\Delta(27)$ has 27 elements divided into 11 equivalence classes. There are 9 one-dimensional representations $\underline{1}_i$ and 2 three-dimensional ones $\underline{3}, \underline{3}'$, with multiplication rules

$$\underline{3} \times \underline{3} = \bar{\underline{3}} + \bar{\underline{3}} + \bar{\underline{3}}, \quad \underline{3} \times \bar{\underline{3}} = \sum_i \underline{1}_i.$$

For the product $\underline{3} \times \underline{3} \times \underline{3}$, there are 3 invariants:

$$\underline{123} + \underline{231} + \underline{312} - \underline{213} - \underline{321} - \underline{132} \quad [SU(3)],$$

$$\underline{123} + \underline{231} + \underline{312} + \underline{213} + \underline{321} + \underline{132} \quad [A_4],$$

$$\text{and } \underline{111} + \underline{222} + \underline{333} \quad [\Delta(27)].$$

If \mathcal{M}_l is diagonal for $(\nu_i, l_i) \sim \underline{\mathbf{3}}, l_i^c \sim \bar{\underline{\mathbf{3}}}, (\phi_i^0, \phi_i^-) \sim \underline{\mathbf{1}}_{1,2,3}$, then

$$\mathcal{M}_\nu = \begin{pmatrix} fa & c & b \\ c & fb & a \\ b & a & fc \end{pmatrix}.$$

Again let $b = c$, then $\Delta m_{sol}^2 \ll \Delta m_{atm}^2$ implies

$$(f+1)a + fb \rightarrow 0, \text{ so that } \tan 2\theta_{12} = \sqrt{2}|f+1|/|f|^2, \\ \Delta m_{atm}^2 = 2|b|^2(|f|^2 - 1)(1 + 2|f|\cos\theta)/|f+1|^2.$$

Hence $|f| \neq 1$ and $|f|\cos\theta \neq -1/2$. For example, if $f = 1.1046$ or $f = -0.5248$, we get $\tan^2 \theta_{12} = 0.45$ and $m_{ee} = 0.05 \text{ eV}$.

Exact **tribimaximal** mixing has also been obtained [**Grimus/Lavoura(2006)**] using the Coxeter group B_4 , which is the symmetry group of the hyperoctahedron (4-crosspolytope) with 384 elements. Here \mathcal{M}_l is diagonal with (ν_i, l_i) , l_i^c , and (ϕ_i^+, ϕ_i^0) belonging to 3 **different** 3-dimensional representations of B_4 , i.e. $a_1 b_1 c_1 + a_2 b_2 c_2 + a_3 b_3 c_3 = \underline{1}$. The \mathcal{M}_N of S_4 is then reduced by $D = E + 3B$.

Other recent studies include

Hagedorn/Lindner/Mohapatra(2006) on S_4 , and **de Medeiros Varzielas/King/Ross(2006)** on $\Delta(27)$.

Conclusion

With the application of the non-Abelian discrete symmetry A_4 , a plausible theoretical understanding of the HPS form of the neutrino mixing matrix has been achieved.

Other symmetries such as S_4 and $\Delta(27)$ are beginning to be studied. They share some of the properties of A_4 and may help to extend our understanding of possible discrete family symmetries, with eventual links to grand unification.